

# Concordance of links with identical Alexander invariants

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**Abstract.** J. Davis showed that the topological concordance class of a link in the 3-sphere is uniquely determined by its Alexander polynomial for 2-component links with Alexander polynomial one. A similar result for knots with Alexander polynomial one was shown earlier by M. Freedman. We prove that these two cases are the only exceptional cases, by showing that the link concordance class is not determined by the Alexander invariants in any other case.

## 1. Introduction

J. Davis proved that if a 2-component link  $L$  has the Alexander polynomial of the Hopf link, namely  $\Delta_L = 1$ , then  $L$  is topologically concordant to the Hopf link [Dav06]. In other words, for 2-components links, the topological concordance class is determined by the Alexander polynomial  $\Delta_L$  when  $\Delta_L = 1$ . A natural question arises from this: *for which links does the Alexander polynomial determine the topological concordance class?*

The answer for knots is already known. A well-known result of M. Freedman (see [Fre84], [FQ90, 11.7B]) says that it holds for Alexander polynomial one knots, and T. Kim [Kim05] (extending earlier work of C. Livingston [Liv02]) showed that it does not hold for any Alexander polynomial which is not one.

The following main result of this note says that the results of M. Freedman and J. Davis are the only cases for which the topological link concordance class is determined by the Alexander polynomial.

**Theorem A.** *Suppose  $L$  is an  $m$ -component link,  $m \geq 2$ , and suppose  $\Delta_L \neq 1$  if  $m = 2$ . Then there are infinitely many links  $L = L_0, L_1, L_2, \dots$  which have the same Alexander polynomial but are mutually not topologically concordant.*

Recall that the multivariable polynomial  $\Delta_L = \Delta_L(x_1, \dots, x_m)$  is well-defined up to multiplication by  $\pm x_1^{a_1} \cdots x_m^{a_m}$ . In particular,  $\Delta_L \neq 1$  means that  $\Delta_L$  is not of the form  $\pm x_1^{a_1} \cdots x_m^{a_m}$ .

We remark that  $\Delta_L \neq 1$  is automatically satisfied for  $m \geq 3$ , as a consequence of the Torres condition, which implies that  $\Delta_L(1, \dots, 1) = 0$  for  $m \geq 3$ . We also remark that in the smooth category, it is known that the conclusion of Theorem A holds even for knots

and 2-component links with Alexander polynomial one. The knot case has been extensively studied in the literature; see, for example, [Gom86, End95] as early works. The case of links with unknotted components has been shown recently in [CKRS12].

In fact, we can say more about the links  $L_i$  in Theorem A. To state the full result, we recall some terminology in the following two paragraphs.

For an  $m$ -component link  $L$  in  $S^3$ , denote its exterior by  $E_L = S^3 - \nu(L)$  where  $\nu(L)$  denotes a tubular neighborhood of  $L$ . We always identify the boundary  $\partial E_L$  with  $m(S^1 \times S^1)$  along the zero-framing, and view  $E_L$  as a bordered 3-manifold with this marking.

The notions of symmetric grope and Whitney tower concordance provide a framework for the study of link concordance. They measure the failure of links to be concordant in terms of fundamental geometric constructions, namely gropes and Whitney towers, in dimension 4. Roughly speaking, one defines a *height  $n$  (symmetric) Whitney tower concordance* by replacing the embedded annuli in the definition of concordance with transversely immersed annuli which form base surfaces supporting a Whitney tower of height  $n$ . A *height  $n$  (symmetric) grope concordance* is defined similarly by replacing annuli with disjointly embedded height  $n$  gropes. These were first used in the context of knot slicing by T. Cochran, K. Orr, and P. Teichner [COT03]. (Detailed definitions for arbitrary links can be found, for example, in [Cha, Section 2.4].) Also, in [Cha, Section 2.3], the first author introduced an analogue of these notions for bordered 3-manifolds, which is called an  $n$ -solvable cobordism. Roughly, an  $n$ -solvable cobordism  $W$  between bordered 3-manifolds  $M$  and  $M'$  is a 4-dimensional cobordism that induces  $H_1(M) \cong H_1(W) \cong H_1(M')$  and admits a certain “lagrangian” with “duals” for the twisted intersection pairing on  $H_2(W; \mathbb{Z}[\pi/\pi^{(n)}])$ , where  $\pi = \pi_1(W)$  and  $\pi^{(n)}$  is the  $n$ th derived subgroup (see Definition 2.3).

We can now state the full version of our main theorem.

**Theorem B.** *Suppose  $L_0$  is an  $m$ -component link, and suppose  $\Delta_L \neq 1$  if  $m = 2$ . Then there are infinitely many links  $L_1, L_2, \dots$  satisfying the following:*

- (1) *For each  $i$ , there is a  $\mathbb{Z}[\mathbb{Z}^m]$ -homology equivalence of  $f: (E_{L_i}, \partial E_{L_i}) \rightarrow (E_{L_0}, \partial E_{L_0})$  rel  $\partial$ , namely  $f|_{\partial}$  is the identification under the zero-framing and*

$$f_*: H_*(E_{L_i}; \mathbb{Z}[\mathbb{Z}^m]) \longrightarrow H_*(E_{L_0}; \mathbb{Z}[\mathbb{Z}^m])$$

*is an isomorphism.*

- (1') *The following invariants are identical for all the  $L_i$ : Alexander polynomial, Alexander ideals, Blanchfield form [Bla57], Milnor's  $\bar{\mu}$ -invariants [Mil57], Orr's transfinite homotopy invariant  $\theta_{\omega}$  [Orr89] (whenever defined), and Levine's homotopy invariant  $\theta$  [Lev89] (whenever defined)  $q$ .*
- (2) *For any  $i \neq j$ , the exteriors  $E_{L_i}$  and  $E_{L_j}$  are not 2-solvably cobordant.*
- (2') *For any  $i \neq j$ , the links  $L_i$  and  $L_j$  are not height 4 grope concordant, not height 4 Whitney tower concordant, and not concordant.*

As references for the Alexander invariants appearing in Theorem B, see, for example, [Kaw96, Hil02].

Experts will easily see that Theorem B (1') and (2') are consequences of (1) and (2), respectively. In Section 2, we discuss this in more details including some background for the reader's convenience.

We remark that the links  $L_i$  in Theorem B can be chosen in such a way that they are indistinguishable to the eyes of the asymmetric Whitney tower/grope theory, which is another framework for the study of link concordance extensively investigated in recent work of J. Conant, R. Schneiderman, and P. Teichner (see [CST11] as an extended summary providing other references). Namely, the  $L_i$  (with the zero-framing) are mutually order  $n$

Whitney tower/grope concordant for any  $n$  in the sense of [CST12, Definition 3.1]. This is discussed in Section 5.

A key ingredient that we use to distinguish concordance classes of links is the Amenable Signature Theorem which first appeared in [CO12]. It generalizes a result presented earlier in the influential work of Cochran-Orr-Teichner [COT03]. In [Cha], the first author formulated a symmetric Whitney tower/grope framework for arbitrary links and bordered 3-manifolds, and gave (a refined version of) the Amenable Signature Theorem as a Cheeger-Gromov  $\rho^{(2)}$ -invariant obstruction to the existence of certain Whitney towers and gropes. In the proof of our main result, we use a special case of this which is stated as Theorem 3.4 in this paper.

An interesting aspect of the proof of Theorem B is that it is separated into two cases which illustrate significantly different aspects contained in the single problem, as discussed below.

For a 3-manifold  $M$ , the Cheeger-Gromov  $\rho^{(2)}$ -invariant  $\rho^{(2)}(M, \phi)$  is a real number associated to a group homomorphism  $\phi: \pi_1(M) \rightarrow \Gamma$ , which we call a representation into  $\Gamma$ . We refer to [CG85, CT07] for details. An essential requirement for the use of  $\rho^{(2)}$ -invariants in the study of concordance and related 4-dimensional equivalence relations is that the representation  $\phi$  should have two properties: (1)  $\phi$  does not annihilate certain interesting elements so that it does not lose too much information, and (2)  $\phi$  factors through the fundamental group of a relevant 4-manifold, for example, the exterior of a concordance, or more generally, a 4-manifold obtained by symmetric surgery on a Whitney tower or a groove.

To find such representations, first we consider links which are “big” in the sense that they admit representations into non-abelian nilpotent quotients. It is straightforward to show (see Lemma 3.2) that a link  $L$  is “big” if and only if  $L$  has either at least 3 components or if  $L$  is a 2-component link  $L$  with  $\text{lk}(L) \neq \pm 1$ . For these links, we apply Dwyer’s theorem to show that representations into certain nilpotent quotients have all the desired properties. (See Theorem 3.1.)

For links which are not big, we employ another approach using the *Blanchfield duality* of the link module  $H_1(E_L; \mathbb{Z}[\mathbb{Z}^m])$ . In fact, for links which admit a nonzero Blanchfield pairing on the torsion part of the link module, this enables us to prove Theorem B using certain representations into *solvable* groups, which are not necessarily nilpotent. (See Theorem 4.1.) This applies especially to 2-component links with  $\text{lk}(L) \neq 0$  and  $\Delta_L \neq 1$ , which we may call “small”. (See Lemma 4.2). The case of small links resembles known approaches to the study of knot concordance [CG86, COT03] and it is related to earlier work of the authors [FP, Cha].

The proofs of the nilpotent and solvable cases of Theorem B occupy Sections 3 and 4 respectively.

We remark that on their own, neither the class of “big” links nor the class of “small” links covers all the cases in Theorem B, while they have a significant overlap, for example two component links  $L$  with  $|\text{lk}(L)| > 1$ . There are links which do not have useful nilpotent representations, for example two component links  $L$  with  $|\text{lk}(L)| = 1$ , so that the Blanchfield pairing method is required as discussed above. On the other hand, there are links for which the Blanchfield pairing method fails to give any useful representations. An enlightening example is the Borromean rings. Its Alexander module is generated by the longitudes. Since the Blanchfield pairing automatically vanishes on the longitudes, it is apparent that the Blanchfield pairing cannot be used to prove property (1) for any representation  $\phi$ .

*Conventions.* Manifolds are assumed to be topological and oriented, and submanifolds are assumed to be locally flat.

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## 2. Some observations on Theorem B

In this section we observe that Theorem B (1') and (2') are consequences of (1) and (2), respectively. We also discuss some necessary background.

Recall that a *link*  $L$  with  $m$  components in  $S^3$  is a union of  $m$  disjoint oriented circles embedded in  $S^3$ . If  $m = 1$  then it is called a *knot*. Two links  $L$  and  $L'$  are *concordant* if there is an  $h$ -cobordism (i.e. disjoint union of annuli) between  $L \times \{0\}$  and  $L' \times \{1\}$  embedded in  $S^3 \times [0, 1]$ .

### Alexander invariants and Blanchfield pairing

We will first discuss the Alexander polynomial, Alexander ideals, and Blanchfield form. The *Alexander module* of a link  $L$  with  $m$  components is defined to be  $H_1(E_L, \{*\}; \mathbb{Z}[\mathbb{Z}^m])$ , viewed as a module over the group ring  $\mathbb{Z}[\mathbb{Z}^m] = \mathbb{Z}[x_1^{\pm 1}, \dots, x_m^{\pm 1}]$ , where the exterior  $E_L$  is endowed with the abelianization map  $\pi_1(E_L) \rightarrow \mathbb{Z}^m$  (sending the  $i$ -meridian to the  $i$ -standard basis vector of  $\mathbb{Z}^m$ ), and  $*$  is a fixed basepoint in  $E_L$ . The module  $H_1(E_L; \mathbb{Z}[\mathbb{Z}^m])$  is called the *link module* of  $L$ . The Alexander polynomial and Alexander ideals are determined by the Alexander module. It is also easy to see from the long exact sequence of a pair that the link module determines the Alexander module and vice versa. Therefore the conclusions in Theorem B (1') on the Alexander polynomials and Alexander ideals are consequences of Theorem B (1).

Let  $Q = \mathbb{Q}(x_1, \dots, x_m)$  be the quotient field of  $\mathbb{Z}[\mathbb{Z}^m]$ , namely the rational function field on  $m$  variables  $x_i$ . For a  $\mathbb{Z}[\mathbb{Z}^m]$ -module  $A$ , we denote its torsion part by

$$tA = \{x \in A \mid rx = 0 \text{ for some nonzero } r \in \mathbb{Z}[\mathbb{Z}^m]\}.$$

Due to Blanchfield [Bla57], there is a sesquilinear pairing

$$tH_1(E_L; \mathbb{Z}[\mathbb{Z}^m]) \times tH_1(E_L; \mathbb{Z}[\mathbb{Z}^m]) \longrightarrow Q/\mathbb{Z}[\mathbb{Z}^m]$$

which is called the *Blanchfield pairing* of  $L$ . It is essentially defined by the duality of  $(E_L, \partial E_L)$  over  $\mathbb{Z}[\mathbb{Z}^m]$ -coefficients. We also refer to [Hil02], particularly Section 2.3, for a thorough discussion of the Blanchfield pairing.

Since it is defined from duality, the Blanchfield pairing is functorial with respect to maps preserving the fundamental class, namely degree one maps on link exteriors. In particular, we have the following: the conclusion of Theorem B (1) that there is a  $\mathbb{Z}[\mathbb{Z}^m]$ -homology equivalence  $f: (E_{L_i}, \partial E_{L_i}) \rightarrow (E_{L_0}, \partial E_{L_0})$  implies that the Blanchfield pairings of  $L_i$  and  $L_0$  are isomorphic. (Note that a  $\mathbb{Z}[\mathbb{Z}^m]$ -homology equivalence is automatically an integral homology equivalence and consequently a degree one map.)

### Milnor's invariants

In [Mil57], J. Milnor defined invariants  $\bar{\mu}_L(I)$  for a link  $L$  with  $m$  components, where  $I$  is a finite sequence of integers in  $\{1, \dots, m\}$ . When  $I$  has length  $|I|$ ,  $\bar{\mu}_L(I)$  is called a  $\bar{\mu}$ -invariant of length  $|I|$ . This is the primary invariant for the study of structure peculiar to link concordance compared to the knot case.

Although  $\overline{\mu}_L(I)$  is originally defined as a certain residue class of an integer, a known method to formulate that “two links have the identical  $\overline{\mu}$ -invariants of length  $\leq q$ ” in the strongest sense is as follows. Recall that the lower central series of a group  $\pi$  is defined by  $\pi_1 := \pi$ ,  $\pi_{q+1} = [\pi, \pi_q]$  where the bracket designates the commutator. We say that *two links  $L$  and  $L'$  with  $\pi = \pi_1(E_L)$  and  $G = \pi_1(E_{L'})$  have the same  $\overline{\mu}$ -invariants of length  $\leq q$*  if there is an isomorphism  $h: \pi/\pi_q \rightarrow G/G_q$  that preserves (the conjugacy class) of each meridian and each 0-linking longitude.

**Lemma 2.1.** *Two links  $L$  and  $L'$  have the same  $\overline{\mu}$ -invariants of any length if there is an integral homology equivalence  $f: (E_L, \partial E_L) \rightarrow (E_{L'}, \partial E_{L'})$  rel  $\partial$ .*

*Proof.* Let  $\pi = \pi_1(E_L)$  and  $G = \pi_1(E_{L'})$ . By Stallings’ theorem [Sta65],  $f$  induces an isomorphism  $h: \pi/\pi_q \cong G/G_q$ . Since  $f$  is fixed on the boundary,  $h$  preserves the conjugacy classes of meridians and longitudes.  $\square$

Since the map  $f$  in the conclusion of Theorem B (1) is automatically an integral homology equivalence, it follows that the Milnor invariant conclusion in Theorem B (1') is a consequence of Theorem B (1).

### Homotopy invariants of Orr and Levine

In [Orr89], K. Orr introduced a homotopy theoretic invariant of links which is still somewhat mysterious. For a link  $L$ , suppose all  $\overline{\mu}$ -invariants vanish. Then for a fixed homomorphism of the free group  $F$  on  $m$  generators into  $\pi = \pi_1(E_L)$  that sends generators to meridians, we obtain an induced isomorphism  $F/F_q \cong \pi/\pi_q$  by Stallings’ theorem [Sta65]. These give rise to  $\pi \rightarrow \overline{F} := \varprojlim F/F_q$  and  $E_L \rightarrow K(\overline{F}, 1)$ . Let  $K_\omega$  be the mapping cone of the map  $K(F, 1) \rightarrow K(\overline{F}, 1)$  induced by the inclusion  $F \rightarrow \overline{F}$ . Then it is easily seen that the map  $E_L \rightarrow K(\overline{F}, 1) \rightarrow K_\omega$  extends to a map  $o_L: S^3 \rightarrow K_\omega$ . Its homotopy class  $\theta_\omega(L) := [o_L] \in \pi_3(K_\omega)$  is Orr’s transfinite homotopy invariant. It is unknown whether this invariant can be non-vanishing for links which have all  $\overline{\mu}$ -invariants zero.

**Lemma 2.2.** *If there is an integral homology equivalence  $f: (E_L, \partial E_L) \rightarrow (E_{L'}, \partial E_{L'})$  rel  $\partial$ , then  $\theta_\omega(L) = \theta_\omega(L')$ .*

*Proof.* Let  $\pi = \pi_1(E_L)$  and  $G = \pi_1(E_{L'})$ . Fix a map  $\mu: F \rightarrow E_L$  sending generators to meridians. The map  $f \circ \mu: F \rightarrow E_{L'}$  also sends generators to meridians. Define the map  $o_L$  and  $o_{L'}: S^3 \rightarrow K_\omega$  as above, using  $\mu$  and  $f \circ \mu$ . From the definition of  $o_L$ , it is easily seen that the map  $g: S^3 \rightarrow S^3$  obtained by filling in  $f$  with the identity map of a solid torus satisfies  $g \circ o_L = o_{L'}$ . Since  $g$  has degree one, it follows that  $\theta_\omega(L) = [o_L] = [o_{L'}] = \theta_\omega(L')$ .  $\square$

This shows that the Orr invariant conclusion in Theorem B (1') is a consequence of Theorem B (1). The same argument works for Levine’s homotopy invariant  $\theta(L)$  defined in [Lev89]. We omit details.

### Solvable cobordism and Whitney tower/grope concordance

The notion of an  $n$ -solvable cobordism used in Theorem B was formulated in [Cha, Section 2.3], as a version of an  $n$ -solution for manifolds with boundary. The notion of an  $n$ -solution was introduced by Cochran-Orr-Teichner [COT03], and was generalized by Harvey [Har08] to the case of links. For later use in this paper we describe its precise definition below. Recall that a bordered 3-manifold  $M$  over a surface  $\Sigma$  is a 3-manifold with boundary identified with  $\Sigma$ , and for two bordered 3-manifolds  $M$  and  $M'$  over the same surface, a relative cobordism  $W$  is a 4-manifold satisfying  $\partial W = M \cup_{\partial} -M'$ .

**Definition 2.3** (Solvable cobordism). We say that a relative cobordism  $W$  between bordered 3-manifolds  $M$  and  $M'$  is an  $n$ -solvable cobordism if (i) the inclusions induce  $H_1(M) \cong H_1(W) \cong H_1(M')$  and (ii) there are homology classes  $\ell_1, \dots, \ell_m, d_1, \dots, d_m \in H_2(W; \mathbb{Z}[\pi/\pi^{(n)}])$ , where  $\pi = \pi_1(W)$ , such that the  $\mathbb{Z}[\pi/\pi^{(n)}]$ -valued intersection pairing  $\lambda_n$  on  $H_2(W; \mathbb{Z}[\pi/\pi^{(n)}])$  satisfies  $\lambda_n(\ell_i, \ell_j) = 0$  and  $\lambda_n(\ell_i, d_j) = \delta_{ij}$ .

In [Cha, Section 2] the following was observed by using techniques in [COT03, Section 8]:

$$\begin{aligned} L \text{ and } L' \text{ are concordant} &\implies L \text{ and } L' \text{ are height } n+2 \text{ grope concordant} \\ &\implies L \text{ and } L' \text{ are height } n+2 \text{ Whitney tower concordant} \\ &\implies E_L \text{ and } E_{L'} \text{ are } n\text{-solvably cobordant} \end{aligned}$$

For the definitions of Whitney tower and grope concordance, refer to, for example, [Cha, Definition 2.12, Definition 2.14].

From the above implications, we see that Theorem B (2') is an immediate consequence of Theorem B (2).

### 3. Links with nontrivial lower central series quotients

The goal of this section is to prove the following special case of Theorem B. Recall that we denote the lower central series of a group  $\pi$  by  $\{\pi_q\}$ .

**Theorem 3.1.** *Suppose  $L$  is an  $m$ -component link with  $\pi = \pi_1(E_L)$  such that  $\pi_2/\pi_3 \neq 0$ . Then there are infinitely many links  $L = L_0, L_1, L_2, \dots$  such that there is a  $\mathbb{Z}[\mathbb{Z}^m]$ -homology equivalence  $f: (E_{L_i}, \partial E_{L_i}) \rightarrow (E_{L_0}, \partial E_{L_0})$  rel  $\partial$  for each  $i$  but the exteriors  $E_{L_i}$  and  $E_{L_j}$  are not 2-solvably cobordant for any  $i \neq j$ .*

From Theorem 3.1 and the discussions in Section 2, it follows that Theorem B holds whenever  $\pi_2/\pi_3 \neq 0$ .

Before we prove Theorem 3.1, we clarify when the lower central series hypothesis is satisfied.

**Lemma 3.2.** *Suppose  $L$  is an  $m$ -component link with  $\pi = \pi_1(E_L)$ ,  $m \geq 2$ .*

- (1) *The abelian group  $\pi_2/\pi_3$  has rank  $\geq (m-1)(m-2)/2$ .*
- (2) *If  $m = 2$ , then  $\pi_2/\pi_3 \cong \mathbb{Z}/\text{lk}(L)\mathbb{Z}$ .*

*Consequently,  $\pi_2/\pi_3 \neq 0$  if and only if either (i)  $m \geq 3$  or, (ii)  $m = 2$  and  $\text{lk}(L) \neq \pm 1$ .*

*Proof.* Milnor [Mil57, Theorem 4] showed that  $\pi/\pi_3$  is presented by

$$\pi/\pi_3 = \langle x_1, \dots, x_m \mid [x_1, \lambda_1], \dots, [x_m, \lambda_m], F_3 \rangle$$

where  $m$  is the number of components of  $L$ ,  $F_3$  is the 3rd lower central subgroup of the free group  $F$  on  $x_1, \dots, x_m$ , and  $\lambda_i$  is an element in  $F$  which represents the  $i$ th longitude of  $L$  in  $\pi/\pi_3$ . It is well known that  $F_2/F_3$  is the free abelian group generated by the basic commutators  $[x_i, x_j]$ ,  $i < j$ , by Hall's basis theorem. Also, we have  $\lambda_i \equiv \prod_{j \neq i} x_j^{\ell_{ij}} \pmod{F_2}$ , where  $\ell_{ij}$  is the linking number of the  $i$ th and  $j$ th components of  $L$ . Using the standard identities  $[a, bc] \equiv [a, b][a, c] \pmod{F_3}$  and  $[a, b]^{-1} = [b, a]$ , we obtain

$$[x_i, \lambda_i] \equiv [x_1, x_i]^{-\ell_{1i}} \cdots [x_{i-1}, x_i]^{-\ell_{(i-1)i}} [x_i, x_{i+1}]^{\ell_{i(i+1)}} \cdots [x_i, x_m]^{\ell_{im}} \pmod{F_3}.$$

From this it follows that  $\pi_2/\pi_3$  is given by the abelian group presentation with  $m(m-1)/2$  generators  $v_{ij} = [x_i, x_j]$ ,  $1 \leq i < j \leq m$ , and the following  $m$  relators for  $i = 1, \dots, m$ :

$$-\ell_{i1}v_{1i} - \cdots - \ell_{(i-1)i}v_{(i-1)i} + \ell_{i(i+1)}v_{i(i+1)} + \cdots + \ell_{im}v_{im} = 0$$

Note that the  $m$  relators add up to zero. Therefore the rank of  $\pi_2/\pi_3$  is at least  $m(m-1)/2 - (m-1) = (m-1)(m-2)/2$ .

For  $m = 2$ , then we have one generator  $v_{12}$  and one relator  $\ell_{12}v_{12} = 0$ . Therefore  $\pi_2/\pi_3 \cong \mathbb{Z}/\ell_{12}\mathbb{Z}$ .  $\square$

In the proof of Theorem 3.1 we will make use of the following definition.

**Definition 3.3.** For a group  $G$  and a sequence  $\mathcal{P} = (R_1, R_2, \dots)$  of commutative rings  $R_i$  with unity, we define the *mixed-coefficient lower central series*  $\{\mathcal{P}_q G\}$  by  $\mathcal{P}_1 G := G$  and

$$\mathcal{P}_{q+1} G := \text{Ker} \left\{ \mathcal{P}_q G \longrightarrow \frac{\mathcal{P}_q G}{[G, \mathcal{P}_q G]} \otimes_{\mathbb{Z}} R_q \right\}.$$

We remark that  $\mathcal{P}_q G$  is a characteristic normal subgroup of  $G$ . We will also make use of the following result from [Cha]:

**Theorem 3.4** (A special case of the Amenable Signature Theorem [Cha, Theorem 3.2]). *Suppose  $W$  is a 2-solvable cobordism between bordered 3-manifolds  $M$  and  $M'$ . Suppose  $\Gamma$  is a group which admits a filtration  $\{e\} \subset \Gamma' \subset \Gamma$  such that  $\Gamma/\Gamma'$  is torsion-free abelian and such that  $\Gamma'$  is either torsion-free abelian or an abelian  $p$ -group for some prime  $p$ . Then for any  $\phi: \pi_1(M \cup_{\partial} -M') \rightarrow \Gamma$  that extends to  $\pi_1(W)$ ,  $\rho^{(2)}(M \cup_{\partial} -M', \phi) = 0$ .*

*Proof.* First note that  $\Gamma$  is a solvable group and therefore amenable. Furthermore it follows from [CO12, Lemma 6.8] that  $\Gamma$  lies in Strebel's class  $D(R)$  [Str74] for  $R = \mathbb{Q}$  or  $R = \mathbb{Z}_p$ , with  $p$  a prime. The theorem is now an immediate consequence of case III of the Amenable Signature Theorem [Cha, Theorem 3.2], since  $\Gamma^{(2)} = \{e\}$  and  $W$  is a 2-solvable cobordism.  $\square$

We are now ready to give the proof of Theorem 3.1. If the reader is interested in link concordance only, then in the proof below the phrase “2-solvable cobordism” can be safely replaced with “concordance exterior.”

*Proof of Theorem 3.1.* By the hypothesis, there exists a simple closed curve  $\alpha$  in  $E_L$  which is a generator of the abelian group  $\pi_2/\pi_3$ . We can and will assume that  $\alpha$  is unknotted in  $S^3$ . We then choose a prime  $p$  which divides the order of  $\alpha$  in  $\pi_2/\pi_3$ ; if  $\alpha$  has infinite order, choose any prime  $p$ .

Let  $\mathcal{P} = (\mathbb{Q}, \mathbb{Z}_p)$ , so that  $\mathcal{P}_q G$  is defined for  $q = 1, 2, 3$ . Then, for not only the given link group  $\pi = \pi_1(E_L)$  but also any  $\pi$  with  $\pi/[\pi, \pi]$  torsion free,  $\mathcal{P}_2 \pi$  is the ordinary lower central subgroup  $\pi_2 = [\pi, \pi]$ . Also,  $\mathcal{P}_2 \pi / \mathcal{P}_3 \pi \cong (\pi_2/\pi_3) \otimes \mathbb{Z}_p$ , a  $\mathbb{Z}_p$ -vector space. Consequently, for the given  $\pi = \pi_1(E_L)$ , our  $\alpha$  represents a nonzero element in  $\mathcal{P}_2 \pi / \mathcal{P}_3 \pi \subset \pi / \mathcal{P}_3 \pi$ , namely an element of order  $p$ .

According to Cheeger and Gromov [CG85, p. 23] there is a constant  $R > 0$  determined by the 3-manifold  $E_L \cup_{\partial} -E_L$  such that  $|\rho^{(2)}(E_L \cup_{\partial} -E_L, \Phi)| < R$  for any homomorphism  $\Phi$ . Let us choose knots  $J_i$  inductively for  $i = 1, 2, \dots$  in such a way that the inequality

$$(1) \quad |\rho^{(2)}(J_i, \mathbb{Z}_p)| > R + |\rho^{(2)}(J_j, \mathbb{Z}_p)|$$

is satisfied whenever  $i > j$ . Here, given a knot  $J$  and  $p \in \mathbb{Z}$  we write

$$\rho^{(2)}(J, \mathbb{Z}_p) := \rho^{(2)}(0\text{-framed surgery on } J, \text{unique epimorphism onto } \mathbb{Z}_p).$$

For example, the connected sum of a sufficiently large number of trefoils can be taken as  $J_i$ . To see this, denote the right handed trefoil by  $T$  and set  $C := \frac{1}{p} \sum_{r=0}^{p-1} \sigma_T(e^{2\pi r \sqrt{-1}/p})$ , where  $\sigma_T(z)$  denotes the Levine-Tristram signature of  $T$  corresponding to  $z \in S^1$ . It is

straightforward to see that  $C > 0$ , since at least one of the  $\sigma_T(z)$  is positive, and all non-zero  $\sigma_T(z)$  have the same sign. If  $J$  is the connected sum of  $k$  copies of  $T$  we see that

$$(2) \quad \rho^{(2)}(J, \mathbb{Z}_p) = \frac{1}{p} \sum_{r=0}^{p-1} \sigma_J(e^{2\pi r \sqrt{-1}/p}) = k \cdot C.$$

Here, for the first equality we appeal to [Fri04, Corollary 4.3] or equivalently [CO12, Lemma 8.7 (2)] and for the second equality we use the additivity of the Levine-Tristram signatures. If we now denote by  $J_i$  the connected sum of  $i \cdot \lceil R/C \rceil$  copies of  $T$ , then (1) is clearly satisfied.

We then use the satellite construction to produce a new link  $L_i := L(\alpha, J_i)$  by tying the knot  $J_i$  into  $L$  along the curve  $\alpha$ . More precisely, by filling in the exterior  $E_\alpha = S^3 - \nu(\alpha)$  with the exterior  $E_{J_i} = S^3 - \nu(J_i)$  along an orientation reversing homeomorphism of the boundary torus  $\partial\nu(\alpha) \rightarrow \partial\nu(J_i)$  that identifies a meridian and 0-linking longitude of  $\alpha$  with a 0-linking longitude and a meridian of  $J_i$  respectively, we obtain a new 3-manifold which is homeomorphic to  $S^3$ , and the image of  $L \subset E_\alpha$  under this homeomorphism is the new link  $L_i = L(\alpha, J_i)$ . We denote a 0-framed push-off of  $\alpha$  in  $E_{L \cup \alpha} \subset E_{L_i}$  by  $\alpha_i$ .

It is well known that there is an integral homology equivalence  $f: (E_{L_i}, \partial E_{L_i}) \rightarrow (E_L, \partial E_L)$  (see e.g. [CO, Lemma 5.3]). In fact  $f$  is obtained by gluing the identity map on  $E_{L \cup \alpha}$  with the standard homology equivalence  $(E_{J_i}, \partial E_{J_i}) \rightarrow (E_{J_0}, \partial E_{J_0}) = S^1 \times (D^2, S^1)$ . Since  $\alpha$  lies in  $[\pi_1(E_L), \pi_1(E_L)]$ , a Mayer-Vietoris argument applied to the above construction shows that  $f$  induces isomorphisms on  $H_*(-; \mathbb{Z}[\mathbb{Z}^m])$ .

By Stallings' theorem [Sta65] and our above discussion on  $\mathcal{P}_q\pi/\mathcal{P}_{q+1}\pi$  for  $q < 3$ , we have an induced isomorphism  $\pi_1(E_{L_i})/\mathcal{P}_3\pi_1(E_{L_i}) \cong \pi/\mathcal{P}_3\pi$ . Since  $f$  restricts to the identity on  $E_{L \cup \alpha}$ , the element  $\alpha_i$  corresponds to  $\alpha$  under this isomorphism.

We will need the following lemma which is a consequence of Dwyer's Theorem [Dwy75], a generalization of Stallings' Theorem. We remark that for the special case of a concordance exterior, Stallings' Theorem can be used instead.

**Lemma 3.5.** *If  $W$  is a 1-solvable cobordism between two bordered 3-manifolds  $M$  and  $M'$  with torsion-free  $H_1(M)$ , then the inclusions induce isomorphisms*

$$\pi_1(M)/\mathcal{P}_3\pi_1(M) \cong \pi_1(W)/\mathcal{P}_3\pi_1(W) \cong \pi_1(M')/\mathcal{P}_3\pi_1(M').$$

*Proof.* Recall Dwyer's theorem [Dwy75]: if  $f: X \rightarrow Y$  induces an isomorphism  $H_1(X) \cong H_1(Y)$  and an epimorphism

$$H_2(X) \longrightarrow H_2(Y)/\text{Im}\{H_2(Y; \mathbb{Z}[\pi_1(W)/\pi_1(W)_q]) \rightarrow H_2(Y)\},$$

then  $f$  induces an isomorphism  $\pi_1(X)_q/\pi_1(X)_{q+1} \cong \pi_1(Y)_q/\pi_1(Y)_{q+1}$ .

In our case, by the definition of an  $n$ -solvable cobordism, we have  $H_1(M) \cong H_1(W) \cong H_1(M')$ . Also, there are 1-lagrangian elements  $\ell_1, \dots, \ell_r$  with 1-duals  $d_1, \dots, d_r$  lying in  $H_2(W; \mathbb{Z}[\pi_1(W)/\pi_1(W)^{(1)}])$  such that the  $\ell_i$  and  $d_j$  generate  $H_2(W)$ . Since  $\pi_1(W)^{(1)}$  is equal to  $\pi_1(W)_2$ , the  $H_2$  condition of Dwyer's theorem is satisfied. Therefore it follows that

$$\pi_1(M)_q/\pi_1(M)_{q+1} \cong \pi_1(W)_q/\pi_1(W)_{q+1} \cong \pi_1(M')_q/\pi_1(M')_{q+1}$$

for  $q = 1, 2$  by Dwyer's theorem. By our observation that  $\mathcal{P}_2\pi = \pi_2$  and  $\mathcal{P}_2\pi/\mathcal{P}_3\pi = \pi_2/\pi_3 \otimes_{\mathbb{Z}} \mathbb{Z}_p$  for groups  $\pi$  with torsion free  $H_1$ , we obtain

$$\mathcal{P}_q\pi_1(M)/\mathcal{P}_{q+1}\pi_1(M) \cong \mathcal{P}_q\pi_1(W)/\mathcal{P}_{q+1}\pi_1(W) \cong \mathcal{P}_q\pi_1(M')/\mathcal{P}_{q+1}\pi_1(M')$$

for  $q = 1, 2$ . From this the desired conclusion follows by the five lemma.  $\square$



Returning to the proof of Theorem 3.1 let  $W$  be a 2-solvable cobordism between  $E_{L_i}$  and  $E_{L_j}$ . We will show that  $i = j$ . First note that we obtain

$$\pi/\mathcal{P}_3\pi \cong \pi_1(E_{L_i})/\mathcal{P}_3\pi_1(E_{L_i}) \cong \pi_1(W)/\mathcal{P}_3\pi_1(W) \cong \pi_1(E_{L_j})/\mathcal{P}_3\pi_1(E_{L_j})$$

by Lemma 3.5. Let  $\phi: \pi_1(W) \rightarrow \Gamma := \pi_1(W)/\mathcal{P}_3\pi_1(W)$  be the projection, and by abuse of notation, we denote its restriction to  $\partial W = E_{L_i} \cup_{\partial} -E_{L_j}$  by  $\phi$  as well. Since  $\alpha$  represents an order  $p$  element in  $\pi/\mathcal{P}_3\pi$ , both  $\phi([\alpha_i])$  and  $\phi([\alpha_j])$  have order  $p$ .

By applying Theorem 3.4 with  $n = 2$ , we obtain that

$$(3) \quad \rho^{(2)}(E_{L_i} \cup_{\partial} -E_{L_j}, \phi) = 0.$$

On the other hand, note that the map  $\phi$  induces a homomorphism  $\varphi: \pi_1(E_L \cup_{\partial} -E_L) \rightarrow \Gamma$  as follows. Recall that  $E_{L_i} \cup_{\partial} -E_{L_j}$  is obtained from  $E_L \cup_{\partial} -E_L$  by satellite constructions using the knots  $J_i$  and  $J_j$ . Viewing  $E_{J_i}$  as a subspace of  $E_{L_i} \cup_{\partial} -E_{L_j}$ , the homomorphism  $\phi$  restricted to  $\pi_1(E_{J_i})$  sends the meridian of  $J_i$  to  $\phi([\alpha_i])$ . Since  $\phi([\alpha_i])$  has order  $p$  in the abelian subgroup  $\mathcal{P}_2\pi/\mathcal{P}_3\pi$  of  $\Gamma$ , it follows that  $\phi$  restricted to  $E_{J_i}$  factors as  $\pi_1(E_{J_i}) \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}_p \hookrightarrow \Gamma$  where the first map is the abelianization. Similarly for  $J_j$ . It follows that  $\phi$  on  $\pi_1(E_{L_i} \cup_{\partial} -E_{L_j})$  gives rise to a homomorphism  $\varphi: \pi_1(E_L \cup_{\partial} -E_L) \rightarrow \Gamma$ . To see this, observe that we can arrange an element  $\gamma \in \pi_1(E_L \cup_{\partial} E_L)$  to avoid  $\nu(\alpha_i)$  and  $\nu(\alpha_j)$ . The image  $\varphi(\gamma)$  can then be defined by  $\phi$ . This is well-defined because crossing  $\alpha_i$  in homotopy of  $\gamma$  changes  $\gamma$  by a meridian of  $\nu(\alpha_i)$ , which in  $E_{L_i}$  is attached to a longitude of  $J_i$ , and therefore maps trivially under  $\phi$ .

Now, using (i) the additivity of  $\rho^{(2)}$  under satellite construction [COT04, Proposition 3.2], (ii) the  $L^2$ -induction property of  $\rho^{(2)}$  [COT03, Proposition 5.13], and (iii) the fact that  $\alpha$  represents an element of order  $p$  in  $\Gamma$ , we obtain that

$$(4) \quad \rho^{(2)}(E_{L_i} \cup_{\partial} -E_{L_j}, \phi) = \rho^{(2)}(E_L \cup_{\partial} -E_L, \varphi) + \rho^{(2)}(J_i, \mathbb{Z}_p) - \rho^{(2)}(J_j, \mathbb{Z}_p).$$

It now follows from (3) and the choice of  $R$  that

$$| |\rho^{(2)}(J_i, \mathbb{Z}_p)| - |\rho^{(2)}(J_j, \mathbb{Z}_p)| | < R.$$

In light of (1) we now see that  $i = j$ .

Thus, when  $i \neq j$ , we have shown that the bordered manifolds  $E_{L_i}$  and  $E_{L_j}$  are not 2-solvably cobordant.  $\square$

#### 4. Links with nontrivial Blanchfield pairing

According to Lemma 3.2 the fundamental group of a 2-component link with linking number equal to  $\pm 1$  admits no non-abelian nilpotent quotients. The goal of this section is to provide an alternative approach, using the Blanchfield duality, to prove Theorem B for such links.

In this section we denote  $\mathbb{Q}[\mathbb{Z}^m]$  by  $\Lambda$ , where  $m$  is understood to be the number of components of the link. The ring  $\mathbb{Q}[\mathbb{Z}^m]$  has the same quotient field as  $\mathbb{Z}[\mathbb{Z}^m]$ , namely the rational function field  $Q = \mathbb{Q}(x_1, \dots, x_m)$  considered in Section 2. Recall that for a  $\Lambda$ -module  $A$  we denote the  $\Lambda$ -torsion submodule by  $tA$ . In what follows  $B\ell_L$  denotes the rational Blanchfield form  $B\ell_L: tH_1(E_L; \Lambda) \times tH_1(E_L; \Lambda) \rightarrow Q/\Lambda$ . We remark that this Blanchfield pairing is obtained by tensoring the integral Blanchfield pairing discussed in Section 2 with  $\mathbb{Q}$ .

**Theorem 4.1.** *Suppose  $L$  is an  $m$ -component link for which the Blanchfield pairing  $B\ell_L$  is not constantly zero, i.e.,  $B\ell_L(x, y) \neq 0$  for some  $x, y \in tH_1(E_L; \Lambda)$ . Then there are infinitely many links  $L = L_0, L_1, L_2, \dots$  such that there is a  $\mathbb{Z}[\mathbb{Z}^m]$ -homology equivalence of  $f: (E_{L_i}, \partial E_{L_i}) \rightarrow (E_{L_0}, \partial E_{L_0})$  rel  $\partial$  for each  $i$  but the exteriors  $E_{L_i}$  and  $E_{L_j}$  are not*

2-solvably cobordant (and consequently the links  $L_i$  and  $L_j$  are not concordant) for any  $i \neq j$ .

Before proving Theorem 4.1, we observe a special case to which Theorem 4.1 applies.

**Lemma 4.2.** *Suppose  $L$  is a 2-component link with  $\text{lk}(L) \neq 0$  and  $\Delta_L \neq 1$ . Then the Blanchfield pairing  $B\ell_L$  is not constantly zero.*

*Proof.* Since  $\text{lk}(L) \neq 0$ , the Blanchfield pairing  $B\ell_L$  on  $H_1(E_L; \Lambda)$  is nondegenerate by Levine [Lev82, Theorem B]. Therefore it suffices to show that  $H_1(E_L; \Lambda)$  is nonzero.

Recall that the Torres condition (see e.g. [Hil02, Section 5.1]) implies that, up to multiplication by a monomial, the following equality holds:

$$\Delta_L(x_1, 1) = (x_1^{|\text{lk}(L)|-1} + \cdots + x_1 + 1)\Delta_K(x_1).$$

Here  $K$  is the first component of  $L$ . In particular we see that  $\Delta_L(1, 1) = |\text{lk}(L)|$ . Our assumptions that  $\text{lk}(L) \neq 0$  and  $\Delta_L \neq 1$  now immediately imply that  $\Delta_L$  is not a monomial. It follows that  $H_1(E_L; \Lambda)$  is nonzero.  $\square$

From Theorem 4.1, Lemma 4.2 and the discussions in Section 2, it follows that Theorem B holds for 2-component links with nonzero linking number and with  $\Delta_L \neq 1$ . This, combined with Theorem 3.1 and Lemma 3.2, completes the proof of Theorem B, modulo the proof of Theorem 4.1 which is given below.

*Proof of Theorem 4.1.* We choose simple closed curves  $\alpha_1, \dots, \alpha_N$  in  $E_L$  which are unknotted in  $S^3$  and have linking number zero with  $L$  such that their classes  $[\alpha_k]$  generate  $tH_1(E_L; \Lambda)$ , which is a finitely generated module since  $\Lambda$  is Noetherian. For each  $i = 1, 2, \dots$ , we use the satellite construction to produce a new link  $L_i = L(\{\alpha_k\}, \{J_{ik}\})$  by tying a collection of knots  $\{J_{ik}\}_{k=1}^N$  into  $L$  along the curves  $\alpha_k$ , for  $k = 1, \dots, N$ . (See the proof of Theorem 3.1 for a more detailed description of the satellite construction.) We define  $J_{0k}$  to be the trivial knot for each  $k$ , so that  $L = L_0 = L(\{\alpha_k\}, \{J_{0k}\})$  is also described in the same way. For  $i \geq 1$  we choose the knots  $J_{ik}$  as follows. As made use of in the proof of Theorem 3.1, due to Cheeger-Gromov [CG85, p. 23], there is a constant  $C > 0$  determined by the 3-manifold  $E_L \cup_{\partial} -E_L$  such that  $|\rho^{(2)}(E_L \cup_{\partial} -E_L, \varphi)| < C$  for any representation  $\varphi$ . For a knot  $K$ , we define  $\rho^{(2)}(K) = \int \sigma_K(\omega) d\omega$ , the integral of the Levine-Tristram signature function  $\sigma_K(\omega)$  over the unit circle normalized to length one. For example, an elementary calculation shows that if  $K$  is the trefoil, then  $\rho^{(2)}(K) = \frac{4}{3}$  (see e.g. [COT03]). We choose knots  $J_i$  inductively for  $i = 1, 2, \dots$  in such a way that the inequality

$$(5) \quad |\rho^{(2)}(J_i)| > C + N|\rho^{(2)}(J_j)|$$

is satisfied whenever  $i > j$ ; recall that  $N$  is the number of satellite curves  $\alpha_k$ . Once again, the connected sum of a sufficiently large number of trefoils can be taken as  $J_i$ . Let  $J_{ik} = J_i$  for every  $k$ .

As we observed in the proof of Theorem 3.1, there is an integral homology equivalence  $f: (E_{L_i}, \partial E_{L_i}) \rightarrow (E_L, \partial E_L)$ . Since each  $\alpha_k$  lies in  $[\pi_1(E_L), \pi_1(E_L)]$ , a Mayer-Vietoris argument applied to our satellite construction shows that  $f$  induces isomorphisms on  $H_*(-; \mathbb{Z}[\mathbb{Z}^m])$ .

Let  $\alpha_{ik} \subset E_{L \cup \alpha} \subset E_{L_i}$  be a push-off of  $\alpha_k$  along the zero-framing. By the above, the  $[\alpha_{ik}]$  generate  $H_1(E_{L_i}; \Lambda) \cong H_1(E_L; \Lambda)$ . As we discussed in Section 2,  $B\ell_{L_i} \cong B\ell_L$  under the map induced by  $f$  since  $f$  is a  $\mathbb{Q}[\mathbb{Z}^m]$ -homology isomorphism.

Suppose  $W$  is a 2-solvable cobordism between  $E_{L_i}$  and  $E_{L_j}$  for some  $i \geq j$ . We will show that this implies that  $i = j$ . We need the following fact, which is obtained immediately by

combining [Cha, Theorem 4.12] with [Cha, Corollary 4.14], the proofs of which relied on arguments due to [COT03, Theorem 4.4 and Lemma 4.5] and [CHL09, Lemma 5.10].

**Theorem 4.3** ([Cha]). *Suppose that  $W$  is a 1-solvable cobordism between link exteriors  $E_L$  and  $E_{L'}$ . Then the submodule  $P = \text{Ker}\{tH_1(E_L; \Lambda) \rightarrow tH_1(W; \Lambda) \subset H_1(W; \Lambda)\}$  satisfies  $B\ell_L(P, P) = 0$ .*

In our case, from Theorem 4.3 and the hypothesis that  $B\ell_L \cong B\ell_{L_i}$  is not constantly zero, it follows that  $P = \text{Ker}\{tH_1(E_{L_i}; \Lambda) \rightarrow tH_1(W; \Lambda)\}$  is not equal to  $tH_1(E_{L_i}; \Lambda)$ . That is,  $[\alpha_{ik}] \notin P$  for some  $k$ .

For a group  $G$ , we now denote by  $\{\mathcal{P}^n G\}$  the rational derived series, namely  $\mathcal{P}^0 G := G$  and

$$\mathcal{P}^{n+1} G := \text{Ker} \left\{ \mathcal{P}^n G \longrightarrow \frac{\mathcal{P}^n G}{[\mathcal{P}^n G, \mathcal{P}^n G]} \otimes_{\mathbb{Z}} \mathbb{Q} \right\}.$$

Then for  $\pi = \pi_1(W)$ ,  $\mathcal{P}^1 \pi$  is the ordinary commutator subgroup  $\pi^{(1)} = [\pi, \pi]$  since the abelian group  $\pi/[\pi, \pi]$  is torsion free. Also,  $\mathcal{P}^1 \pi / \mathcal{P}^2 \pi$  is the quotient of  $\pi^{(1)} / \pi^{(2)} = H_1(W; \mathbb{Z}[\mathbb{Z}^m])$  by its  $\mathbb{Z}$ -torsion subgroup.

Let  $\phi: \pi \rightarrow \Gamma := \pi / \mathcal{P}^2 \pi$  be the projection, and by abuse of notation, we denote its restriction to  $\partial W = E_{L_i} \cup_{\partial} -E_{L_j}$  by  $\phi$  as well. Since  $\mathcal{P}^1 \pi / \mathcal{P}^2 \pi = H_1(W; \mathbb{Z}[\mathbb{Z}^2]) / (\mathbb{Z}\text{-torsion})$  injects into  $H_1(W; \Lambda)$  and the image of  $[\alpha_{ik}]$  in  $H_1(W; \Lambda)$  is nontrivial for some  $k$ , it follows that  $\phi([\alpha_{ik}])$  is nontrivial for some  $k$ . Furthermore,  $\phi([\alpha_{ik}])$  has infinite order, since  $\phi([\alpha_{ik}])$  lies in  $\mathcal{P}^1 \pi / \mathcal{P}^2 \pi$  which is a torsion-free abelian group. Similarly  $\phi([\alpha_{jk}])$  has infinite order for some (possibly different)  $k$ .

Theorem 3.4 applies to the group  $\Gamma$ , with  $\Gamma' = \mathcal{P}^1 \pi / \mathcal{P}^2 \pi$ . We thus deduce that  $\rho^{(2)}(E_{L_i} \cup_{\partial} -E_{L_j}, \phi) = 0$ .

The homomorphism  $\phi$  on  $\pi_1(E_{L_i} \cup_{\partial} -E_{L_j})$  gives rise to a homomorphism  $\varphi: \pi_1(E_L \cup_{\partial} -E_L) \rightarrow \Gamma$ , which is defined in a very similar way to the map which was also called  $\varphi$  in the proof of Theorem 3.1: the homomorphism  $\phi$  restricted to  $\pi_1(E_{J_{ik}})$  sends the meridian of  $J_{ik}$  to  $\phi([\alpha_{ik}])$ . Since  $\phi([\alpha_{ik}])$  has infinite order in the abelian subgroup  $\mathcal{P}^1 \pi / \mathcal{P}^2 \pi$  of  $\Gamma$ , it follows that  $\phi$  restricted to  $E_{J_{ik}}$  factors as  $\pi_1(E_{J_i}) \rightarrow \mathbb{Z} \hookrightarrow \Gamma$  where the first map is the abelianization. Similarly for  $J_{jk}$ . It follows that  $\phi$  on  $\pi_1(E_{L_i} \cup_{\partial} -E_{L_j})$  gives rise to a homomorphism of  $\pi_1(E_L \cup_{\partial} -E_L)$ , say  $\varphi$ .

By the above and by the same argument as in the proof of Theorem 3.1 we obtain that

$$\begin{aligned} 0 &= \rho^{(2)}(E_{L_i} \cup_{\partial} -E_{L_j}, \phi) \\ (6) \quad &= \rho^{(2)}(E_L \cup_{\partial} -E_L, \varphi) + \sum_{k=1}^N \epsilon_{ik} \rho^{(2)}(J_{ik}, \psi) - \sum_{k=1}^N \epsilon_{jk} \rho^{(2)}(J_{jk}, \psi) \end{aligned}$$

where  $\psi$  denotes the abelianization epimorphism of a knot group onto  $\mathbb{Z}$  and where  $\epsilon_{ik}$  is 0 if  $\phi([\alpha_{ik}])$  is trivial, 1 otherwise, and  $\epsilon_{ij}$  is defined similarly. Furthermore, by [COT04, Proposition 5.1] we know that for any knot  $K$  the invariant  $\rho^{(2)}(K, \psi)$  is equal to the integral of the Levine-Tristram signature function. Since  $\epsilon_{ik} = 1$  for some  $k$  (and similarly for  $\epsilon_{jk}$ ) and our  $J_i$  are chosen so that the inequality (5) is satisfied whenever  $i > j$ , the equality (6) can only be satisfied if  $i = j$ . From this it follows that there is no 2-solvable cobordism between  $E_{L_i}$  and  $E_{L_j}$  whenever  $i \neq j$ .  $\square$

## 5. Satellite construction and asymmetric Whitney towers

In this section we observe that our links  $L_i$  in Theorem B can be assumed to be mutually order  $n$  Whitney tower/grope concordant for any  $n$ . For the definition of order  $n$  Whitney

tower concordance of framed links, see [CST12, Definition 3.2]; in this section we assume that links are always endowed with the zero-framing.

Since we constructed  $L_i$  by satellite construction on a given link  $L$  using some knots which are only required to have sufficiently large integral (or average of finitely many evaluations) of the Levine-Tristram signature, the claim follows immediately from the following lemma:

**Lemma 5.1.** *Suppose  $K$  is a knot in  $S^3$  with vanishing Arf invariant,  $L$  is a link in  $S^3$ , and  $\alpha$  is a simple closed curve in  $S^3 - L$  which is unknotted in  $S^3$ . Then the link  $L' = L(\alpha, K)$  obtained by the satellite construction is order  $n$  Whitney tower/grope concordant to  $L$  for any  $n$ .*

For example, in the construction of our examples above, take an even number of trefoils for  $J_i$ . Note that asymmetric Whitney tower/grope concordance contains no information even when we use representations to nilpotent groups to obstruct symmetric Whitney tower concordance.

We remark that in [Sch06], Schneiderman showed that the Whitney tower and grope concordance are equivalent in the asymmetric case. Therefore it suffices to show our results for grope concordance. A brief outline of the proof is:  $K$  bounds an order  $n$  grope in  $D^4$  since  $\text{Arf}(K) = 0$ , and then a “boundary connected sum” of parallel copies of this grope and the product concordance from  $L$  to  $L$  becomes an order  $n$  grope concordance between  $L$  and  $L'$ . The details are spelled out below.

*Proof of Lemma 5.1.* We begin with a well-known description of the satellite construction. Choose an embedded 2-disk  $D$  in  $S^3$  which is bounded by  $\alpha$  and meets  $L$  transversely. Choose an open regular neighborhood  $U$  of  $D$  in  $S^3$  for which  $(U, U \cap L)$  is a trivial  $r$ -string link where  $r = |D \cap L|$ . For the knot  $K$ , take the union  $Y$  of  $r$  parallel copies of  $K$  and take an open regular neighborhood  $V$  of a 2-disk fiber of the normal bundle of  $K$  such that  $(V, V \cap Y)$  is a trivial  $r$ -string link. There is an orientation reversing homeomorphism  $h: (U, U \cap L) \rightarrow (V, V \cap Y)$  such that

$$(S^3, L') = \left( ((S^3, L) - (U, U \cap L)) \cup ((S^3, Y) - (V, V \cap Y)) \right) / x \sim h(x) \text{ for } x \in \partial U.$$

Here components of  $Y - V$  are oriented according to the sign of the intersection points in  $D \cap L$ .

From our assumption that  $\text{Arf}(K) = 0$  and the result in [Sch05] that the Arf invariant is the only obstruction to a knot bounding a framed embedded grope of arbitrary order, it follows that there is a framed embedded grope of order  $n$  in  $D^4$  bounded by  $K$  for any  $n$ . Taking  $r$  parallel copies of the grope (and orienting the base surfaces according to the sign of the intersection points in  $D \cap L$ ), we obtain a framed embedded grope  $G$  bounded by  $Y$ .

Identify a collar neighborhood of  $S^3$  in  $D^4$  with  $S^3 \times [0, \epsilon]$ . We may assume that  $V \times [0, \epsilon]$  intersects (the base surface of)  $G$  in  $(V \cap Y) \times [0, \epsilon]$ . Now, define  $G' \subset S^3 \times [0, 1]$  by forming the union

$$(S^3 \times [0, 1], G') = \left( ((S^3, L) \times [0, 1] - (U, U \cap L) \times [0, \epsilon]) \cup ((D^4, G) - (V, V \cap Y) \times [0, \epsilon]) \right) / \sim$$

where  $(x, t) \sim h(x, t)$  for  $(x, t) \in (\partial U \times [0, \epsilon]) \cup (U \times \{\epsilon\})$ . Then  $G'$  is a grope concordance of order  $n$  cobounded by  $L' \times 0$  and  $L \times 1$ .  $\square$

## References

- [Bla57] Richard C. Blanchfield, *Intersection theory of manifolds with operators with applications to knot theory*, Ann. of Math. (2) **65** (1957), 340–356.
- [CG85] Jeff Cheeger and Mikhael Gromov, *Bounds on the von Neumann dimension of  $L^2$ -cohomology and the Gauss-Bonnet theorem for open manifolds*, J. Differential Geom. **21** (1985), no. 1, 1–34.
- [CG86] Andrew Casson and Cameron Gordon, *Cobordism of classical knots*, À la recherche de la topologie perdue, Birkhäuser Boston, Boston, MA, 1986, With an appendix by P. M. Gilmer, pp. 181–199.
- [Cha] Jae Choon Cha, *Symmetric Whitney tower cobordism for bordered 3-manifolds and links*, arXiv:1204.4968, to appear in Trans. Amer. Math. Soc.
- [CHL09] Tim D. Cochran, Shelly Harvey, and Constance Leidy, *Knot concordance and higher-order Blanchfield duality*, Geom. Topol. **13** (2009), no. 3, 1419–1482.
- [CKRS12] Jae Choon Cha, Taehee Kim, Daniel Ruberman, and Saso Strle, *Smooth concordance of links topologically concordant to the Hopf link*, Bull. Lond. Math. Soc. **44** (2012), no. 3, 443–450.
- [CO] Jae Choon Cha and Kent E. Orr, *Hidden torsion, 3-manifolds, and homology cobordism*, arXiv:1101.4092, to appear in J. Topol.
- [CO12] ———,  *$L^2$ -signatures, homology localization, and amenable groups*, Comm. Pure Appl. Math. **65** (2012), 790–832.
- [COT03] Tim D. Cochran, Kent E. Orr, and Peter Teichner, *Knot concordance, Whitney towers and  $L^2$ -signatures*, Ann. of Math. (2) **157** (2003), no. 2, 433–519.
- [COT04] ———, *Structure in the classical knot concordance group*, Comment. Math. Helv. **79** (2004), no. 1, 105–123.
- [CST11] Jim Conant, Rob Schneiderman, and Peter Teichner, *Higher-order intersections in low-dimensional topology*, Proc. Natl. Acad. Sci. USA **108** (2011), no. 20, 8131–8138.
- [CST12] James Conant, Rob Schneiderman, and Peter Teichner, *Whitney tower concordance of classical links*, Geom. Topol. **16** (2012), no. 3, 1419–1479.
- [CT07] Tim D. Cochran and Peter Teichner, *Knot concordance and von Neumann  $\rho$ -invariants*, Duke Math. J. **137** (2007), no. 2, 337–379.
- [Dav06] James F. Davis, *A two component link with Alexander polynomial one is concordant to the Hopf link*, Math. Proc. Cambridge Philos. Soc. **140** (2006), no. 2, 265–268.
- [Dwy75] William G. Dwyer, *Homology, Massey products and maps between groups*, J. Pure Appl. Algebra **6** (1975), no. 2, 177–190.
- [End95] Hisaaki Endo, *Linear independence of topologically slice knots in the smooth cobordism group*, Topology Appl. **63** (1995), no. 3, 257–262.
- [FP] Stefan Friedl and Mark Powell, *Links not concordant to the Hopf link*, arXiv:1105.2773, to appear in Math. Proc. Camb. Phil. Soc.
- [FQ90] Michael H. Freedman and Frank Quinn, *Topology of 4-manifolds*, Princeton Mathematical Series, vol. 39, Princeton University Press, Princeton, NJ, 1990.
- [Fre84] Michael H. Freedman, *The disk theorem for four-dimensional manifolds*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Warsaw, 1983) (Warsaw), PWN, 1984, pp. 647–663.
- [Fri04] Stefan Friedl, *Eta invariants as sliceness obstructions and their relation to Casson-Gordon invariants*, Algebr. Geom. Topol. **4** (2004), 893–934 (electronic).
- [Gom86] Robert E. Gompf, *Smooth concordance of topologically slice knots*, Topology **25** (1986), no. 3, 353–373.
- [Har08] Shelly Harvey, *Homology cobordism invariants and the Cochran-Orr-Teichner filtration of the link concordance group*, Geom. Topol. **12** (2008), 387–430.
- [Hil02] Jonathan Hillman, *Algebraic invariants of links*, Series on Knots and Everything, vol. 32, World Scientific Publishing Co. Inc., River Edge, NJ, 2002.
- [Kaw96] Akio Kawauchi, *A survey of knot theory*, Birkhäuser Verlag, Basel, 1996, Translated and revised from the 1990 Japanese original by the author.
- [Kim05] Taehee Kim, *An infinite family of non-concordant knots having the same Seifert form*, Comment. Math. Helv. **80** (2005), no. 1, 147–155.
- [Lev82] Jerome P. Levine, *The module of a 2-component link*, Comment. Math. Helv. **57** (1982), no. 3, 377–399.
- [Lev89] ———, *Link concordance and algebraic closure. II*, Invent. Math. **96** (1989), no. 3, 571–592.

- [Liv02] Charles Livingston, *Seifert forms and concordance*, Geom. Topol. **6** (2002), 403–408 (electronic).
- [Mil57] John W. Milnor, *Isotopy of links. Algebraic geometry and topology*, A symposium in honor of S. Lefschetz, Princeton University Press, Princeton, N. J., 1957, pp. 280–306.
- [Orr89] Kent E. Orr, *Homotopy invariants of links*, Invent. Math. **95** (1989), no. 2, 379–394.
- [Sch05] Rob Schneiderman, *Simple Whitney towers, half-gropes and the Arf invariant of a knot*, Pacific J. of Math. **222** (2005), no. 1, 169–184.
- [Sch06] ———, *Whitney towers and gropes in 4-manifolds*, Trans. Amer. Math. Soc. **358** (2006), no. 10, 4251–4278 (electronic).
- [Sta65] John Stallings, *Homology and central series of groups*, J. Algebra **2** (1965), 170–181.
- [Str74] Ralph Strebel, *Homological methods applied to the derived series of groups*, Comment. Math. Helv. **49** (1974), 302–332.